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ON THE COMPARISON OF STABLE AND UNSTABLE p-COMPLETION

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Abstract. In this note we show that a p-complete nilpotent space X has a p-complete suspension spectrum if and only if its homotopy groups $\pi_* X$ are bounded p-torsion. In contrast, if $\pi_* X$ is not all bounded p-torsion, we locate uncountable rational vector spaces in the integral homology and in the stable homotopy groups of X. To prove this, we establish a homological criterion for p-completeness of connective spectra. Moreover, we illustrate our results by studying the stable homotopy groups of $K(\mathbb{Z}_p, n)$ via Goodwillie calculus.

1. Introduction

The notion of p-completion plays a fundamental role in algebra and topology, for it provides effective means to isolate and study p-primary properties. Applied to homotopy theory by Bousfield and Kan [BK72] as well as Sullivan [Sul74] and developed further in [Bou75, Bou79], it has since become one of the standard tools in the hands of algebraic topologists. However, there appears to be no general account of the comparison between unstable and stable p-completion in the literature, which is the question we address in the present note.

Our main goal is to characterize p-complete spaces which have p-complete suspension spectra:

Theorem 4.7. *If X is a p-complete nilpotent space, then $\Sigma^\infty X$ is p-complete if and only if $\pi_n X$ is bounded p-torsion for each n.*

In fact, we exhibit a sharp dichotomy of p-complete nilpotent spaces: if X is a p-complete nilpotent space whose homotopy groups are not all bounded p-torsion, then the integral homology groups and stable homotopy groups of X both contain an uncountable rational vector space. As a consequence, we deduce that a nilpotent space X with derived p-complete integral homology and unstable homotopy must have both $H_n(X;\mathbb{Z})$ and $\pi_n X$ of bounded p-torsion for all n.

In a first step towards the proof of the theorem, we complement the second author's characterization of p-complete spectra in terms of homotopy groups with an integral homological criterion, using a mild generalization of Serre classes appropriate for stable homotopy theory. This is in sharp contrast to the aforementioned fact that the integral homology of p-complete spaces is not well-behaved, and thus cannot be used to characterize p-completeness of spaces.

Corollary 3.3. *A bounded below spectrum X is p-complete if and only if $H_*(X;\mathbb{Z})$ is derived p-complete in each degree.*

In order to use this result to prove the theorem, we need to detect rational classes in the homology of p-complete spaces whose homotopy is not bounded p-torsion. This rests on the study of the integral homology of p-complete spheres. We end this note with a sample computation, illustrating how Goodwillie calculus allows us to detect rational classes in the stable homotopy groups of the Eilenberg–MacLane space $K(\mathbb{Z}_p, n)$.

Proposition 5.3. *For $n \geq 1$ and $k > 1$, the stable homotopy group $\pi_{nk}\Sigma^\infty K(\mathbb{Z}_p, n)$ contains an uncountable rational vector space. In particular, $\Sigma^\infty K(\mathbb{Z}_p, n)$ is not p-complete.*

In fact, we also give a short alternative argument based on the integral homology of $K(\mathbb{Z}_p, n)$.

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Conventions. Throughout this paper, p will be a fixed prime number and $\mathbb{Z}_p$ denotes the p-adic integers. We say that a nilpotent group N is bounded p-torsion if there exists an m such that for all $x \in N$, we have $x^{p^m} = 1$. A graded nilpotent group N_* is said to be of bounded p-torsion if N_k is bounded p-torsion for each k; however, we do not require a uniform bound. Whenever we are in a graded context, we indicate the degree of an abelian group A by square brackets, i.e., $A[n]$ refers to A placed in degree n. If X is a topological space, then $H_*(X; A)$ is the reduced homology of X with coefficients in A. For a space or spectrum X, we write $\tau_{\leq n}X = \tau_{<n+1}X$ for the n-th Postnikov section of X and $\tau_{\geq n+1}X = \tau_{>n}X$ for the fiber of the canonical map $X \to \tau_{\leq n}X$.

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2. Preliminaries on p-completion

We briefly recall the basic properties of p-completion for nilpotent groups, topological spaces, and spectra. With the exceptions of Lemma 2.2 and Proposition 2.4, this material is mostly taken from [BK72, Bou75, Bou79], and we refer to these sources as well as [HS99, MP12] for further references.

2.1. **Algebraic p-completion for abelian groups.** In general, the p-completion functor $M \mapsto \lim_i M/p^iM$ on the category of abelian groups is neither left nor right exact, so one studies its zeroth and first left derived functors L_0 and L_1, which may be expressed as $L_0M = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M)$ and $L_1M = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, M)$ by [BK72, Ch. VI, 2.1]. An abelian group M is called derived p-complete (or Ext-p-complete or L-complete) if the natural completion map $M \to L_0M$ is an isomorphism. For each abelian group M, the map $M \to L_0M$ will then be the universal homomorphism from M to a derived p-complete abelian group by [BK72, Ch. VI, 3.2]. We will denote the full subcategory of derived p-complete abelian groups by $\mathcal{C}_p$.

Proposition 2.1. *The category $\mathcal{C}_p$ is a full abelian subcategory of $\mathrm{Mod}_{\mathbb{Z}}$ closed under extensions and limits. Furthermore, for any $M \in \mathrm{Mod}_{\mathbb{Z}}$ there is a short exact sequence*

$$0 \longrightarrow \lim^1_i \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^i, M) \longrightarrow L_0M \longrightarrow \lim_i M/p^iM \longrightarrow 0$$

relating derived p-completion to ordinary p-completion.

Proof. This is essentially proven in [BK72, Ch. VI, 2.1], but can also be deduced as a special case of [HS99, Thms. A.2 and A.6]. □

We will later make use of the following observation.

Lemma 2.2. *If $A \in \mathcal{C}_p$ is torsion, then A is bounded p-torsion.*

Proof. We give two proofs, a conceptual one and an elementary argument. First, any derived p-complete group A has $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[1/p], A) = 0 = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}[1/p], A)$ by [BK72, Ch. VI, 3.4], and hence A satisfies $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, A) = 0 = \mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Q}, A)$ since $\mathbb{Q}$ is a quotient of free $\mathbb{Z}[1/p]$-modules. Thus, A is a cotorsion group with no nontrivial divisible subgroups, so the Baer–Fomin theorem [Bae36] implies that A is a bounded p-torsion group.

Second, suppose that the conclusion of the lemma is false, i.e., that there exists a sequence $(a_i)_{i\in\mathbb{N}}$ of elements of A such that the order of a_i is p^i. Set $x_j = \sum_{i=0}^{j-1} a_{2i+1}p^i$, then the element

$x = (x_1, x_2, x_3, \ldots) \in \prod_{j\in\mathbb{N}} A$ lies in $\lim_j A/p^j$. By construction, x is not p-torsion, which contradicts the fact that $A \to \lim_j A/p^j$ is surjective, forcing $\lim_j A/p^j$ to be p-torsion. □

Remark 2.3. By a theorem of Prüfer, the conclusion of the lemma implies that A must in fact be a direct sum of cyclic p-groups.

2.2. **Algebraic p-completion for nilpotent groups.** Recall from [BK72, Ch. VI, §2] that the notion of derived p-completion can be extended to nilpotent groups, as follows: If $X^\wedge_p$ denotes the Bousfield–Kan p-completion of a nilpotent space X as recalled in the next subsection, then we define the derived p-completion of the nilpotent group N as $L_0N = \pi_1(K(N,1)^\wedge_p)$ and $L_1N = \pi_2(K(N,1)^\wedge_p)$. A nilpotent group N is called derived p-complete if the completion map $N \to L_0N$ is an isomorphism; for each nilpotent group N, the map $N \to L_0N$ will then be the universal homomorphism from N to a derived p-complete nilpotent group by [BK72, Ch. VI, 3.2]. We denote the category of derived p-complete nilpotent groups by $\mathcal{N}_p$.

The inclusion functor $\mathcal{C}_p \to \mathcal{N}_p$ has a left adjoint given by taking a derived p-complete nilpotent group N to the derived p-completion of its abelianization $L_0(N/[N,N])$. Note that the unit of this adjunction is surjective, i.e., for any derived p-complete nilpotent group N, the canonical map $N \to L_0(N/[N,N])$ is surjective. Indeed, since L_0 preserves epimorphisms of nilpotent groups, all maps in the following commutative diagram are surjective:

$$\begin{tikzcd} N \arrow[r] \arrow[d, "\cong"'] & N/[N,N] \arrow[d] \\ L_0N \arrow[r] & L_0(N/[N,N]). \end{tikzcd}$$

We obtain the following generalization of Lemma 2.2:

Proposition 2.4. *The following conditions are equivalent for $N \in \mathcal{N}_p$:*

(1) N is torsion.
(2) $L_0(N/[N,N])$ is torsion.
(3) N is bounded p-torsion.

Proof. The surjectivity of the map $N \to L_0(N/[N,N])$ observed above immediately gives the implication (1) ⇒ (2), while (3) ⇒ (1) is trivial.

Assume that $L_0(N/[N,N])$ is torsion and thus bounded p-torsion by Lemma 2.2. Consider the lower central series of N,

$$N = \gamma_1 N \supseteq \gamma_2 N \supseteq \ldots \supseteq \gamma_m N = 1,$$

with successive abelian quotients $Q_i(N) = \gamma_i N/\gamma_{i+1}N$. We claim that, for each $i \geq 1$, $Q_i(N)$ is a direct sum of a p-divisible group and a bounded p-torsion group. Indeed, we start with the abelianization $Q_1(N) = N/[N,N]$ of N. Lemma 3.7 in [BK72, Ch. VI] implies that the kernel of the completion map $Q_1(N) \to L_0Q_1(N)$ is p-divisible, so the claim holds for $Q_1(N)$. The general case follows from this, because $\bigoplus_{i\geq 1} Q_i(N)$ is generated as a Lie algebra by $Q_1(N)$. By [BK72, Ch. VI, 2.5], there is an exact sequence

$$L_0Q_i(N) \longrightarrow L_0(N/\gamma_{i+1}N) \longrightarrow L_0(N/\gamma_i N) \longrightarrow 1$$

for any $i \geq 1$. Using the previous claim, $L_0Q_i(N)$ is bounded p-torsion, so we see inductively that $L_0(N/\gamma_i N)$ is bounded p-torsion for all $i \geq 1$, hence (3) holds. □

Remark 2.5. The implication (1) ⇒ (3) in the previous proposition could also be proven more directly via the upper central series of N, whose quotients are known to be derived p-complete by [BK72, VI. 3.4(ii)], but this result would be insufficient for our later use.

2.3. **Topological p-completion.** In [BK72], Bousfield and Kan introduced the notion of p-completion for topological spaces, lifting the algebraic notion defined above to topology. In general, the p-completion of a space is difficult to describe, but the theory simplifies significantly for nilpotent spaces; in particular, in this case p-completion coincides with $H\mathbb{F}_p$-localization [Bou75]. Furthermore, for nilpotent spaces with $\mathbb{F}_p$-homology of finite type, p-completion can be identified with p-profinite completion due to Sullivan [Sul74]. Similarly, the category of spectra admits (at least) two notions of p-completion, given either by $H\mathbb{F}_p$-localization or, the one we will use here, localization at the mod p Moore spectrum S^0/p, see [Bou79]. The next result summarizes the relation between these constructions and lists their basic properties.

Theorem 2.6 (Bousfield, Kan)**.**

(1) A nilpotent space X is p-complete if and only if $\pi_n X$ is derived p-complete for all $n \in \mathbb{N}$. Moreover, the notions of p-completion and $H\mathbb{F}_p$-localization coincide for nilpotent spaces.

(2) A spectrum X is p-complete if and only if $\pi_n X$ is derived p-complete for all $n \in \mathbb{Z}$. If X is bounded below, then X is p-complete if and only if X is $H\mathbb{F}_p$-local.

Moreover, if X is a nilpotent space or spectrum, then there exists a splittable short exact sequence computing the unstable or stable homotopy groups of its p-completion, respectively:

$$0 \longrightarrow L_0\pi_n X \longrightarrow \pi_n(X_p^\wedge) \longrightarrow L_1\pi_{n-1}X \longrightarrow 0$$

for any n, where $L_i(-) \cong \operatorname{Ext}^{1-i}_{\mathbb{Z}}(\mathbb{Z}/p^\infty, -)$ are the derived functors of p-completion.

3. Generalized Serre theory

The full subcategory $\mathcal{C}_p$ of $\mathrm{Mod}_{\mathbb{Z}}$ is not closed under subobjects or quotients, and thus does not form a Serre class in the usual sense. This necessitates a mild generalization of Serre's mod $\mathcal{C}$ theory which we develop in this section.

Definition 3.1. A weak Serre class is a full subcategory $\mathcal{C} \subseteq \mathrm{Mod}_{\mathbb{Z}}$ such that if

$$A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow A_4 \longrightarrow A_5$$

is an exact sequence in $\mathrm{Mod}_{\mathbb{Z}}$ with $A_1, A_2, A_4, A_5 \in \mathcal{C}$, then also $A_3 \in \mathcal{C}$.

More explicitly, this means that $\mathcal{C} \subseteq \mathrm{Mod}_{\mathbb{Z}}$ is a full additive subcategory closed under kernels, cokernels, and extensions. It follows that $\mathcal{C}$ is also closed under tensoring and $\operatorname{Tor}_1^{\mathbb{Z}}$ with respect to finitely generated abelian groups. For instance, any Serre subcategory of $\mathrm{Mod}_{\mathbb{Z}}$ is a weak Serre class, but the converse does not hold. The main example of interest to us here is the category $\mathcal{C}_p$ of derived p-complete abelian groups, see Proposition 2.1.

Proposition 3.2. *Suppose $\mathcal{C}$ is a weak Serre class. If X is a bounded below spectrum, then the following two conditions are equivalent:*

(1) $\pi_n X \in \mathcal{C}$ for all $n \in \mathbb{Z}$.

(2) $H_n(X;\mathbb{Z}) \in \mathcal{C}$ for all $n \in \mathbb{Z}$.

Proof. Assume the first condition holds; we will argue via the Postnikov tower $(\tau_{\leq n}X)$ of X. For simplicity, we will write $H_*(Y)$ for the integral homology of a spectrum Y throughout this proof.

To start with, we need to show that $H_*(HA) \in \mathcal{C}$ for $A \in \mathcal{C}$. Using the isomorphisms $H_*(HA) \cong H_*(H\mathbb{Z}; A)$, the universal coefficient theorem gives a short exact sequence

$$0 \longrightarrow H_*(H\mathbb{Z}) \otimes_{\mathbb{Z}} A \longrightarrow H_*(HA) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(H_{*-1}(H\mathbb{Z}), A) \longrightarrow 0.$$

In each degree, the integral Steenrod algebra $H_*(H\mathbb{Z})$ is finitely generated over $\mathbb{Z}$, as follows from Serre theory for the class of finitely generated abelian groups. Therefore, the outer terms of this sequence are in $\mathcal{C}$. This shows $H_*(HA) \in \mathcal{C}$ as well.

Given $n \in \mathbb{Z}$, we will now prove that $H_n(X) \in \mathcal{C}$. Since $H_n(\tau_{>n}X) = 0 = H_{n-1}(\tau_{>n}X)$ by connectivity, we see that $H_n(X) \cong H_n(\tau_{\leq n}X)$. This reduces the claim to proving that $H_*(\tau_{\leq n}X) \in \mathcal{C}$. This follows inductively, using the exact sequence

$$H_{*+1}(\tau_{\leq n-1}X) \longrightarrow H_*(\Sigma^n H\pi_n X) \longrightarrow H_*(\tau_{\leq n}X) \longrightarrow H_*(\tau_{\leq n-1}X) \longrightarrow H_{*-1}(\Sigma^n H\pi_n X)$$

associated to the fiber sequence $\Sigma^n H(\pi_n X) \to \tau_{\leq n}X \to \tau_{\leq n-1}X$. Since $H_k(H\pi_n X) \in \mathcal{C}$ for all $k \in \mathbb{Z}$, this gives the implication (1) $\Rightarrow$ (2).

For the converse, consider the convergent Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} \cong H_s(X; \pi_t S^0) \implies \pi_{s+t}X.$$

Since $\pi_t S^0$ is finitely generated over $\mathbb{Z}$ for each $t \in \mathbb{Z}$, $H_s(X; \pi_t S^0) \in \mathcal{C}$ for each bidegree (s,t), hence $\pi_n X$ is also in $\mathcal{C}$ for all $n \in \mathbb{Z}$. □

When applied to the weak Serre class $\mathcal{C}_p$, we obtain a homological characterization of p-completeness for bounded below spectra.

Corollary 3.3. *For a bounded below spectrum X, the following conditions are equivalent:*

(1) X is p-complete.
(2) $\pi_n X$ is derived p-complete for all n.
(3) $H_n(X; \mathbb{Z})$ is derived p-complete for all n.

Proof. The equivalence of (1) and (2) is the content of Theorem 2.6(2), while (2) is equivalent to (3) by Proposition 3.2. □

We deduce that the integral homology of p-complete spaces is well-behaved in the stable range.

Corollary 3.4. *Suppose X is p-complete space. If X is n-connected, then $H_k(X; \mathbb{Z})$ is derived p-complete for all $k \leq 2n$.*

Proof. Since $\pi_k \Sigma^\infty X \cong \pi_k X$ for $k \leq 2n$ by the Freudenthal suspension theorem, Theorem 2.6 implies that $\pi_* \tau_{\leq 2n} \Sigma^\infty X$ is derived p-complete in each degree, hence so is $H_*(\tau_{\leq 2n}\Sigma^\infty X; \mathbb{Z})$ by Corollary 3.3. We thus get that $H_k(X; \mathbb{Z}) \cong H_k(\Sigma^\infty X; \mathbb{Z}) \cong H_k(\tau_{\leq 2n}\Sigma^\infty X; \mathbb{Z})$ is derived p-complete for $k \leq 2n$. □

Corollary 3.5. *For a bounded below spectrum X, there exists a splittable short exact sequence computing the integral homology groups of its p-completion:*

$$0 \longrightarrow L_0 H_n(X; \mathbb{Z}) \longrightarrow H_n(X^\wedge_p; \mathbb{Z}) \longrightarrow L_1 H_{n-1}(X; \mathbb{Z}) \longrightarrow 0$$

for any n.

Proof. Since the spectrum $H\mathbb{Z} \wedge X^\wedge_p$ is p-complete by Corollary 3.3, there is a canonical map $(H\mathbb{Z} \wedge X)^\wedge_p \to H\mathbb{Z} \wedge X^\wedge_p$, and this map must be an equivalence because it is an $H\mathbb{F}_p$-equivalence of p-complete bounded below spectra. Hence, the claim follows from Theorem 2.6. □

From Corollary 3.5, we obtain the following description of the p-complete sphere spectrum as a Moore spectrum.

Example 3.6. There is a canonical equivalence $S^0_p \xrightarrow{\sim} M\mathbb{Z}_p$.

4. The comparison

In this section, we first study the relation between p-completion for spectra and spaces under the infinite loop space functor Ω^∞, and then prove our main theorem.

4.1. **Infinite loop spaces.** It is easy to deduce from Theorem 2.6 the following relation between unstable and stable p-completion under Ω^∞.

Proposition 4.1. *For 0-connected spectra X and Y, we have:*

1. *X is p-complete if and only if $\Omega^\infty X$ is p-complete.*
2. *A map $f\colon X \to Y$ is an $H\mathbb{F}_p$-equivalence if and only if $\Omega^\infty f$ is an $H\mathbb{F}_p$-equivalence.*
3. *The canonical comparison map $(\Omega^\infty X)^\wedge_p \to \Omega^\infty(X^\wedge_p)$ is an equivalence.*

Proof. Since $\pi_*\Omega^\infty X \cong \pi_* X$ and $\Omega^\infty X$ is nilpotent, the first claim is a direct consequence of Theorem 2.6. In order to prove (2), note that f is an $H\mathbb{F}_p$-equivalence if and only if the homotopy groups $\pi_* \operatorname{cof}(f)$ of the cofiber of f are uniquely p-divisible. This is equivalent to the statement that the $\mathbb{F}_p$-homology $H_*(\Omega^\infty \operatorname{cof}(f); \mathbb{F}_p)$ is trivial. The Serre spectral sequence associated to the fiber sequence

$$\Omega^\infty X \xrightarrow{\Omega^\infty f} \Omega^\infty Y \longrightarrow \Omega^\infty \operatorname{cof}(f)$$

thus shows that this happens if and only if $\Omega^\infty f$ is an $H\mathbb{F}_p$-equivalence.

Statement (1) implies that $\Omega^\infty(X^\wedge_p)$ is p-complete, so the map $\Omega^\infty(X) \to \Omega^\infty(X^\wedge_p)$ factors canonically through $\phi\colon (\Omega^\infty X)^\wedge_p \to \Omega^\infty(X^\wedge_p)$, making the following diagram commute:

$$\begin{tikzcd} \Omega^\infty X \arrow[r] \arrow[dr] & (\Omega^\infty X)^\wedge_p \arrow[d] \\ & \Omega^\infty(X^\wedge_p). \end{tikzcd}$$

By Statement (2), both the horizontal and the diagonal map are $H\mathbb{F}_p$-equivalences, hence so is the vertical comparison map. □

Remark 4.2. Let Ω^∞_0 be the 0-component of Ω^∞. The last part of the proposition can be strengthened to an equivalence $(\Omega^\infty_0 X)^\wedge_p \to \Omega^\infty_0(X^\wedge_p)$ for any connective spectrum X such that $\pi_0 X$ does not contain any copies of $\mathbb{Z}/p^\infty$. To prove this directly, one may use the short exact sequences displayed at the end of Theorem 2.6.

4.2. **Suspension spectra.** We now turn to the comparison under Σ^∞. In odd dimensions, the next result has also been observed in [BK72, Rem. VI.5.7], see also [MP12, Rem. 11.1.5].

Lemma 4.3. *Let $n \geq 1$ and write S^n_p for the p-completion of S^n. There exists an uncountable rational vector space in $H_{2n}(S^n_p;\mathbb{Z})$ which injects into $H_{2n}(K(\mathbb{Z}_p,n);\mathbb{Z})$ under the map $S^n_p \to \tau_{\leq n} S^n_p \simeq K(\mathbb{Z}_p, n)$.*

Proof. Consider the following segment of the Serre long exact sequence for the fibration $F \to S^n_p \to K(\mathbb{Z}_p, n)$:

$$H_{2n}(F;\mathbb{Z}) \longrightarrow H_{2n}(S^n_p;\mathbb{Z}) \longrightarrow H_{2n}(K(\mathbb{Z}_p,n);\mathbb{Z}) \longrightarrow H_{2n-1}(F;\mathbb{Z}) \longrightarrow \dots.$$

Corollary 3.4 implies that $H_{2n}(F;\mathbb{Z})$ and $H_{2n-1}(F;\mathbb{Z})$ are derived p-complete. Recalling that $\operatorname{Hom}_\mathbb{Z}(\mathbb{Q}, A) = 0 = \operatorname{Ext}^1_\mathbb{Z}(\mathbb{Q}, A)$ whenever A is derived p-complete, we see that the natural map $\operatorname{Hom}_\mathbb{Z}(\mathbb{Q}, H_{2n}(S^n_p;\mathbb{Z})) \to \operatorname{Hom}_\mathbb{Z}(\mathbb{Q}, H_{2n}(K(\mathbb{Z}_p,n);\mathbb{Z}))$ is surjective. Thus, it will suffice to show that $H_{2n}(K(\mathbb{Z}_p,n);\mathbb{Z})$ contains an uncountable rational vector space, which will be verified in the homological proof of Proposition 5.3 below. □

Note that, because $H_*(S^n_p;\mathbb{F}_p) \cong H_*(S^n;\mathbb{F}_p) \cong \mathbb{F}_p[n]$, an application of the universal coefficient theorem shows that $H_k(S^n_p;\mathbb{Z})$ is rational for all $k > n$.

Lemma 4.4. *Suppose N is a derived p-complete nilpotent (abelian) group and $n = 1$ ($n \geq 1$). If N is not bounded p-torsion, then there exists an element $x \in N$ of infinite order inducing a monomorphism $H_*(K(\mathbb{Z}_p, n); \mathbb{Q}) \to H_*(K(N, n); \mathbb{Q})$.*

Proof. By assumption on N and Proposition 2.4, $L_0(N/[N, N])$ contains elements of infinite order. Let $\overline{x}$ be such an element and let $x \in N$ be a lift of $\overline{x}$. For the remainder of the proof we assume $n = 1$; the (easier) case $n \geq 2$ and N abelian is proven similarly. The element x induces a map

$$K(\mathbb{Z}_p, 1) \longrightarrow K(N, 1) \longrightarrow K(L_0(N/[N, N]), 1)$$

such that the composite is injective on π_1. It follows that the rationalization $K(\mathbb{Z}_p, 1)_{\mathbb{Q}} \to K(L_0(N/[N, N]), 1)_{\mathbb{Q}}$ of this map is split, hence the composite

$$H_*(K(\mathbb{Z}_p, 1); \mathbb{Q}) \longrightarrow H_*(K(N, 1); \mathbb{Q}) \longrightarrow H_*(K(L_0(N/[N, N]), 1); \mathbb{Q})$$

is a split monomorphism, which implies the claim. □

Proposition 4.5. *If X is a p-complete nilpotent space whose homotopy groups are not all bounded p-torsion, then the integral homology groups $H_*(X; \mathbb{Z})$ and the stable homotopy groups $\pi_* \Sigma^\infty X$ both contain an uncountable rational vector space.*

Proof. Assume that $\pi_* X$ is not all bounded p-torsion, and let $\pi_n X$ be the lowest such group. It then follows from Lemma 4.4 that $\pi_n X$ contains a class x of infinite order inducing a monomorphism $H_*(K(\mathbb{Z}_p, n); \mathbb{Q}) \to H_*(K(\pi_n X, n); \mathbb{Q})$. Since the map $\tau_{\geq n} X \to X$ is a rational homology equivalence, any rational subgroup of $H_*(\tau_{\geq n} X; \mathbb{Z})$ must map monomorphically to $H_*(X; \mathbb{Z})$, so it suffices to prove the homological claim for $\tau_{\geq n} X$. The element x yields a map $S^n_p \to \tau_{\geq n} X$ such that the composite $S^n_p \to \tau_{\geq n} X \to K(\pi_n X, n)$ factors as

$$\begin{tikzcd}
\tau_{\geq n} X \arrow[r] & \tau_{\leq n} \tau_{\geq n} X \simeq K(\pi_n X, n) \\
S^n_p \arrow[u] \arrow[r] & \tau_{\leq n} S^n_p \simeq K(\mathbb{Z}_p, n). \arrow[u]
\end{tikzcd}$$

It follows from Lemma 4.3 and the choice of x that the induced homomorphism in homology

$$H_{2n}(S^n_p; \mathbb{Z}) \longrightarrow H_{2n}(\tau_{\geq n} X; \mathbb{Z}) \longrightarrow H_{2n}(K(\pi_n X, n); \mathbb{Z})$$

maps an uncountable rational vector space monomorphically to $H_{2n}(K(\pi_n X, n); \mathbb{Z})$, hence so does the map $H_{2n}(S^n_p; \mathbb{Z}) \to H_{2n}(\tau_{\geq n} X; \mathbb{Z})$. This verifies the claim about the integral homology of X.

Recall that, for any connective spectrum Y, the Hurewicz map $\pi_* Y \to H_*(Y; \mathbb{Z})$ has kernel and cokernel of bounded torsion in each degree. Indeed, the fiber sequence $Y \wedge \tau_{>0} S^0 \to Y \to Y \wedge H\mathbb{Z}$ reduces this claim to showing that $\pi_*(Y \wedge \tau_{>0} S^0)$ is bounded torsion in each degree. This follows from the convergent Atiyah–Hirzebruch spectral sequence

$$H_s(Y; \pi_t \tau_{>0} S^0) \implies \pi_{s+t}(Y \wedge \tau_{>0} S^0),$$

because $H_s(Y; \pi_t \tau_{>0} S^0)$ is bounded torsion for all s and t. Therefore, any rational vector space in $H_*(Y; \mathbb{Z})$ may be lifted back to $\pi_* Y$. In particular, an uncountable rational vector space in $H_{2n}(X; \mathbb{Z})$ may be lifted back to $\pi_{2n}(\Sigma^\infty X)$ after suspension. □

Remark 4.6. Suppose X is a p-complete nilpotent space such that $\pi_n X$ is the lowest homotopy group not of bounded p-torsion. The above argument shows that $H_{2n}(X; \mathbb{Z})$ contains an uncountable rational vector space. With more work, we can also show that $H_k(X; \mathbb{Z})$ is derived p-complete for $k \leq 2n - 2$ and thus cannot contain any rational classes. Note that when X is

$(n-1)$-connected, this follows immediately from Corollary 3.4 since $H_k(X;\mathbb{Z})$ is in the stable range.

We can now prove our main theorem.

Theorem 4.7. *If X is a p-complete nilpotent space, then $\Sigma^\infty X$ is p-complete if and only if $\pi_n X$ is bounded p-torsion for each n.*

Note that the torsion exponent of $\pi_n X$ may vary with n and does not need to be bounded uniformly for all n.

Proof. First assume that X is a p-complete nilpotent space with $\pi_n X$ of bounded p-torsion for each n; we can apply [BK72, Ch. II, 4.7] to see that the Postnikov tower of X can be refined to a tower of principal fibrations whose fibers are Eilenberg–MacLane spaces for bounded p-torsion abelian groups. The category of bounded p-torsion abelian groups forms a Serre class, so Serre theory implies that $H_*(X;\mathbb{Z}) \cong H_*(\Sigma^\infty X;\mathbb{Z})$ is degreewise bounded p-torsion. Hence, $\Sigma^\infty X$ is p-complete as a spectrum by Corollary 3.3.

The converse is a consequence of Proposition 4.5: if $\pi_* X$ is not all bounded torsion, then $H_*(\Sigma^\infty X;\mathbb{Z})$ contains rational classes and thus cannot be derived p-complete, hence $\Sigma^\infty X$ is not p-complete by Corollary 3.3. □

The next result generalizes [PSS17, Prop. 2.4].

Corollary 4.8. *If X is a pointed connected space with degreewise finite homotopy groups, then the canonical map $(\Sigma^\infty X)^\wedge_p \to \Sigma^\infty X^\wedge_p$ is an equivalence.*

Proof. By [BK72, Ch. VII, 4.3], X is a $\mathbb{Z}/p$-good space and $X^\wedge_p$ is a p-complete nilpotent space whose homotopy groups are all finite p-groups. Hence $\Sigma^\infty X^\wedge_p$ is p-complete by Theorem 4.7. It follows that the natural map $(\Sigma^\infty X)^\wedge_p \to \Sigma^\infty X^\wedge_p$ is an $H\mathbb{F}_p$-equivalence between $H\mathbb{F}_p$-local spectra, which implies the claim. □

Corollary 4.9. *If X is a nilpotent space with $H_n(X;\mathbb{Z})$ and $\pi_n X$ derived p-complete for all n, then $H_n(X;\mathbb{Z})$ and $\pi_n X$ are bounded p-torsion for all n.*

Proof. The assumption on $\pi_* X$ implies that X is p-complete by Theorem 2.6, while the assumption on $H_*(X;\mathbb{Z})$ shows that $\Sigma^\infty X$ is p-complete, using Corollary 3.3. It thus follows from Theorem 4.7 that $\pi_* X$ is degreewise bounded p-torsion, hence so is $H_*(X;\mathbb{Z})$ by the proof of Theorem 4.7. □

The analogue of this corollary does not hold stably, as the following example demonstrates.

Example 4.10. Let $M(\mathbb{Z}_p, n)$ be the Moore space for $\mathbb{Z}_p$ in degree $n \geq 2$. As $H_*(\Sigma^\infty M(\mathbb{Z}_p,n);\mathbb{Z})$ is isomorphic to $\mathbb{Z}_p[n]$, we see that $\Sigma^\infty M(\mathbb{Z}_p,n)$ is p-complete and consequently has derived p-complete stable homotopy groups and integral homology groups. However, $H_n(\Sigma^\infty M(\mathbb{Z}_p,n);\mathbb{Z}) \cong \mathbb{Z}_p$ is clearly not bounded p-torsion. In particular, $M(\mathbb{Z}_p,n)$ is not p-complete, so this also shows that the assumption that X be p-complete cannot be dropped in Theorem 4.7.

5. Rational classes in the stable homotopy groups of $K(\mathbb{Z}_p, n)$

In this section, we present an example that illustrates how the rational classes in the stable homotopy groups of p-complete spaces arise. In fact, we present two different approaches: One using the integral homology of $K(\mathbb{Z}_p,n)$, and one using Goodwillie calculus. The latter derivation is entirely stable and might be of independent interest.

First, we need a well-known auxiliary result; we outline a proof because we were unable to find a published reference for it. For an abelian group A and any $k \geq 0$, let $\mathrm{Sym}^k_{\mathbb{Z}}(A)$ and $\Lambda^k_{\mathbb{Z}}(A)$ be the kth symmetric power and the kth exterior power on A, respectively.

Lemma 5.1. *If $k > 1$, then $\Lambda^k_{\mathbb{Z}}(\mathbb{Z}_p)$ and the kernel of the multiplication map $\mathrm{Sym}^k_{\mathbb{Z}}(\mathbb{Z}_p) \to \mathbb{Z}_p$ are uncountable rational vector spaces.*

Proof. Since both symmetric and exterior power commute with base-change along $\mathbb{Z} \to \mathbb{Z}/l$ for any prime l, the indicated maps are isomorphisms mod l. Moreover, $\mathrm{Sym}^k_{\mathbb{Z}}(A)$ and $\Lambda^k_{\mathbb{Z}}(A)$ are torsion-free whenever A is, so both $\ker(\mathrm{Sym}^k_{\mathbb{Z}}(\mathbb{Z}_p) \to \mathbb{Z}_p)$ and $\Lambda^k_{\mathbb{Z}}(\mathbb{Z}_p)$ are rational vector spaces. We may therefore base-change to $\mathbb{Q}$, where it is easy to verify that the $\mathbb{Q}$-dimension of the groups under consideration is that of $\mathbb{Q}_p$. □

Remark 5.2. A similar argument also shows that $\mathbb{Z}_p/\mathbb{Z}_{(p)}$ is a rational vector space with the same $\mathbb{Q}$-dimension as $\mathbb{Q}_p$.

Proposition 5.3. *For $n \geq 1$ and all $k > 1$, the stable homotopy group $\pi_{nk}\Sigma^\infty K(\mathbb{Z}_p, n)$ contains an uncountable rational vector space. In particular, $\Sigma^\infty K(\mathbb{Z}_p, n)$ is not p-complete.*

First proof. Let A be an abelian group and recall that $H_*(K(A, n); \mathbb{Z})$ equipped with the Pontryagin product is a graded commutative algebra such that squares of odd dimensional elements are zero; in fact, it has the structure of a graded divided power algebra, see [EML54, Car56] or more recently [Ric09]. With notation as in the previous lemma, the canonical isomorphism $A \to H_n(K(A, n); \mathbb{Z})$ thus extends to a natural homomorphism

$$\begin{cases} \phi^k(A, n) \colon \Lambda^k_{\mathbb{Z}}(A) \longrightarrow H_{kn}(K(A, n); \mathbb{Z}), & \text{if } n \text{ odd} \\ \phi^k(A, n) \colon \mathrm{Sym}^k_{\mathbb{Z}}(A) \longrightarrow H_{kn}(K(A, n); \mathbb{Z}), & \text{if } n \text{ even} \end{cases}$$

for any $n, k > 0$. Moreover, we know that $\phi^k(A, n) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a rational isomorphism. It then follows from Lemma 5.1 that, for $k > 1$, there exists an uncountable rational vector space which is mapped monomorphically to $H_{kn}(K(\mathbb{Z}_p, n); \mathbb{Z})$ via $\phi^k(\mathbb{Z}_p, n)$. We thus obtain an uncountable rational vector space in $H_{kn}(K(\mathbb{Z}_p, n); \mathbb{Z})$ that may be lifted back to give the desired uncountable rational vector space in $\pi_{nk}\Sigma^\infty K(\mathbb{Z}_p, n)$ for $k > 1$, as in the proof of Proposition 4.5. □

Second proof. We will compute the homotopy groups of $\Sigma^\infty K(\mathbb{Z}_p, n) \simeq \Sigma^\infty\Omega^\infty\Sigma^n H\mathbb{Z}_p$ using Goodwillie calculus [Goo03]. To this end, recall that the Goodwillie tower $(P_k)_{k\geq 1}$ associated to the functor $\Sigma^\infty\Omega^\infty \colon \mathrm{Sp} \to \mathrm{Sp}$ is assembled from fiber sequences of functors

$$D_k \longrightarrow P_k \longrightarrow P_{k-1} \tag{5.4}$$

with layers $D_k X \simeq X^{\wedge k}_{h\Sigma_k}$, where the homotopy orbits are formed with respect to the permutation action of Σ_k (see for example [KM13] and the references given therein). Moreover, the Goodwillie tower $(P_k)_{k\geq 0}$ converges for connective spectra, i.e., there is a canonical equivalence

$$\Sigma^\infty\Omega^\infty X \xrightarrow{\sim} \lim_k P_k X$$

for any connective $X \in \mathrm{Sp}$. We will apply this in the case $X = \Sigma^n H\mathbb{Z}_p$.

In order to understand the layers, we start by analyzing $\pi_*(\Sigma^n H\mathbb{Z}_p)^{\wedge k}$ via the universal coefficient theorem. We claim that, for all $k \geq 1$, the homotopy groups have the following form

$$\pi_*(\Sigma^n H\mathbb{Z}_p)^{\wedge k} \cong \begin{cases} 0 & * < nk \\ \mathbb{Z}_p^{\otimes_{\mathbb{Z}} k} & * = nk \\ \text{finite} & * > nk. \end{cases} \tag{5.5}$$

By the universal coefficient theorem, we have an isomorphism

$$\pi_*(\Sigma^n H\mathbb{Z}_p)^{\wedge k} \cong (\pi_*(\Sigma^n H\mathbb{Z})^{\wedge k}) \otimes_{\mathbb{Z}} \mathbb{Z}_p^{\otimes k}.$$

In degrees $* > nk$, the groups $\pi_*(\Sigma^n H\mathbb{Z})^{\wedge k}$ are torsion, so the only torsion-free summand appears in degree nk. Since $\pi_*(\Sigma^n H\mathbb{Z})^{\wedge k}$ is finitely generated over $\mathbb{Z}$ in each degree, the claim follows.

We now plug the formula (5.5) into the convergent homotopy orbit spectral sequence

$$H_s(\Sigma_k, \pi_t(\Sigma^n H\mathbb{Z}_p)^{\wedge k}) \implies \pi_{s+t} D_k(\Sigma^n H\mathbb{Z}_p).$$

There are two cases: If $t > nk$ or $t < nk$, then the groups $H_s(\Sigma_k, \pi_t(\Sigma^n H\mathbb{Z}_p)^{\wedge k})$ are finite or trivial for all s, respectively. Let $t = nk$. By Lemma 5.1 and (5.5), there is an isomorphism $H_s(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)^{\wedge k}) \cong H_s(\Sigma_k, \mathbb{Z}_p)$ for $s > 0$ and $H_0(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)^{\wedge k})$ contains an uncountable rational vector space V_k if $k > 1$. To see the last statement, it suffices to compute the coinvariants on the rational submodule of $\mathbb{Z}_p^{\otimes_{\mathbb{Z}} k}$ by choosing a $\mathbb{Q}$-bases, as in the proof of Lemma 5.1. Furthermore, since the integral homology of Σ_k is finitely generated over $\mathbb{Z}$ in each degree and rationally trivial in positive degrees, $H_s(\Sigma_k, \pi_{nk}(\Sigma^n H\mathbb{Z}_p)^{\wedge k})$ is finite for all $s > 0$. Combining all this information, we obtain $D_1 \Sigma^n H\mathbb{Z}_p \simeq \Sigma^n H\mathbb{Z}_p$ and for $k > 1$:

$$\pi_* D_k(\Sigma^n H\mathbb{Z}_p) \cong \begin{cases} 0 & * < nk \\ V_k \oplus W_k & * = nk \\ \text{finite} & * > nk, \end{cases} \tag{5.6}$$

where V_k is an uncountable rational vector space and W_k is some abelian group.

This allows us to derive a structural formula for $\pi_* P_k \Sigma^n H\mathbb{Z}_p$. Consider the following segment of the long exact sequence of homotopy groups associated to the fiber sequence (5.4):

$$\ldots \longrightarrow \pi_{nk+1} P_{k-1} \Sigma^n H\mathbb{Z}_p \longrightarrow \pi_{nk} D_k \Sigma^n H\mathbb{Z}_p \longrightarrow \pi_{nk} P_k \Sigma^n H\mathbb{Z}_p \longrightarrow \ldots .$$

Because $n \geq 1$, it follows inductively from (5.6) that the term on the left is finite, hence V_k must be a summand in $\pi_{nk} P_k \Sigma^n H\mathbb{Z}_p$. This yields for all $k \geq 1$:

$$\pi_* P_k \Sigma^n H\mathbb{Z}_p \cong \begin{cases} 0 & * < n \\ V_l \oplus W'_l & * = nl \text{ with } 1 \leq l \leq k \\ \text{finite} & \text{otherwise}, \end{cases} \tag{5.7}$$

where V_l is as above for $l \geq 2$, and V_1 and W'_l are some abelian groups.

Finally, since $D_k \Sigma^n H\mathbb{Z}_p$ is nk-connective for all k, the tower $(\pi_* P_k \Sigma^n H\mathbb{Z}_p)_{k \geq 0}$ stabilizes after finally many steps in each degree and hence is Mittag-Leffler. The corresponding Milnor sequence thus degenerates to an isomorphism

$$\pi_* \Sigma^\infty K(\mathbb{Z}_p, n) \cong \pi_* \Sigma^\infty \Omega^\infty \Sigma^n H\mathbb{Z}_p \cong \lim_k \pi_* P_k \Sigma^n H\mathbb{Z}_p.$$

Therefore, the claim follows from (5.7). □

References

[Bae36] Reinhold Baer. The subgroup of the elements of finite order of an abelian group. *Ann. of Math. (2)*, 37(4):766–781, 1936. (cit. on p. 2).

[BK72] A. K. Bousfield and D. M. Kan. *Homotopy limits, completions and localizations*. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972. (cit on pp. 1, 2, 3, 4, 6, 8).

[Bou75] A. K. Bousfield. The localization of spaces with respect to homology. *Topology*, 14:133–150, 1975. (cit on pp. 1, 2, 4).

[Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979. (cit on pp. 1, 2, 4).

[Car56] Henri Cartan. *Séminaire Henri Cartan de l'Ecole Normale Supérieure, 1954/1955. Algèbres d'Eilenberg-MacLane et homotopie*. Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1956. 2ème éd. (cit. on p. 9).

[EML54] Samuel Eilenberg and Saunders Mac Lane. On the groups $H(\Pi, n)$. II. Methods of computation. *Ann. of Math. (2)*, 60:49–139, 1954. (cit. on p. 9).

[Goo03] Thomas G. Goodwillie. Calculus. III. Taylor series. *Geom. Topol.*, 7:645–711, 2003. (cit. on p. 9).

[HS99] Mark A Hovey and Neil P Strickland. Morava K-theories and localisation. *Mem. Am. Math. Soc.*, 139(666):viii+100–100, 1999. (cit. on p. 2).

[KM13] Nicholas Kuhn and Jason McCarty. The mod 2 homology of infinite loopspaces. *Algebr. Geom. Topol.*, 13(2):687–745, 2013. (cit. on p. 9).

[MP12] J. P. May and K. Ponto. *More concise algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2012. Localization, completion, and model categories. (cit on pp. 2, 6).

[PSS17] S. Precht Reeh, T. M. Schlank, and N. Stapleton. A formula for p-completion by way of the Segal conjecture. *ArXiv e-prints*, April 2017. (cit. on p. 8).

[Ric09] Birgit Richter. Divided power structures and chain complexes. In *Alpine perspectives on algebraic topology*, volume 504 of *Contemp. Math.*, pages 237–254. Amer. Math. Soc., Providence, RI, 2009. (cit. on p. 9).

[Sul74] Dennis Sullivan. Genetics of homotopy theory and the Adams conjecture. *Ann. of Math. (2)*, 100:1–79, 1974. (cit on pp. 1, 4).

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